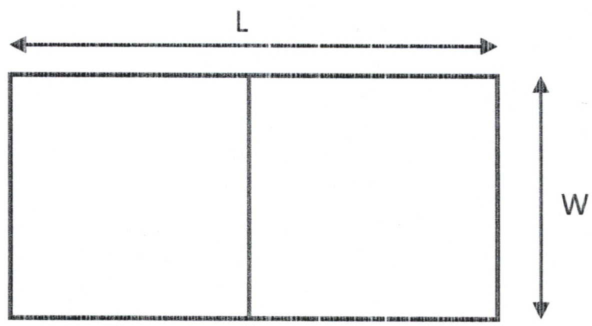
Quadratic equations, minimum/maximum, word problems.

1. Question type: Min (or Max) optimization question

You have a 1200-foot roll of fencing and a large field. You want to make two paddocks by splitting a rectangular enclosure in half. What are the dimensions of the largest such enclosure?



You will see many max/min problems again in calculus.

Fence length: (*) $2 \cdot L + 3 \cdot W = 1200$

$2L = 1200 - 3W \Rightarrow \underline{L = 600 - 1.5W}$

(**) $\text{Area} = L \cdot W$

Now, substitute: $\text{Area} = (600 - 1.5W) \cdot W = \underline{\overbrace{-1.5}^{a} W^2 + \overbrace{600}^{b} W}$

Vertex: $\boxed{W} = \dfrac{-b}{2a} = \dfrac{-600}{2 \cdot (-1.5)} = \boxed{200}$ ft

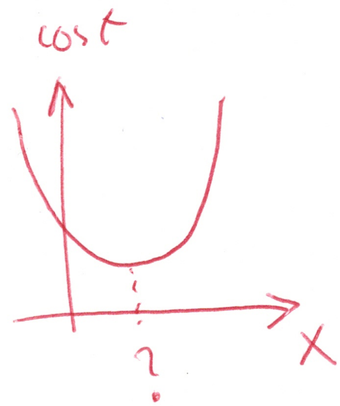
Putting back into (*): $\boxed{L} = 600 - 1.5 \times 200$
$= \boxed{300}$ ft

2. **Question type: Given the formula, and just plug-in**

Your factory produces lemon-scented widgets. You know that as each unit is cheaper, the more you produce. But you also know that costs will eventually go up if you make too many widgets, due to the costs of storage of the overstock. The guy in accounting says that your cost for producing x thousands of units a day can be approximated by the formula

$$C = 0.04x^2 - 8.504x + 25302$$

Find the daily production level that will minimize your costs.



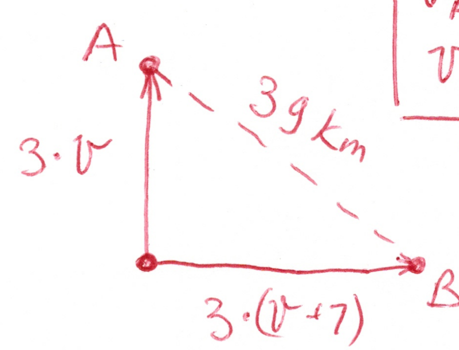
$$X_{vertex} = \frac{-b}{2a} = \frac{8.504}{0.08} = 106.3$$

Number of units to make: $\boxed{106{,}300}$ units

3. **Question type: Pythagorean**

Cyclists A and B leave the same point at the same time, biking at right angle to each other. B travels 7km/h FASTER than A. After 3 hours they are 39km apart. Find the speed of each.

$$\boxed{\begin{array}{l} v_A = 5\ \frac{km}{h} \\ v_B = 12\ \frac{km}{h} \end{array}}$$



$$(3v)^2 + [3\cdot(v+7)]^2 = 39^2$$

$$9v^2 + 9(v^2 + 14v + 49) = 39^2$$

$$v^2 + v^2 + 14v + 49 = 169$$

$$2v^2 + 14v - 120 = 0$$

$$v^2 + 7v - 60 = 0$$

$$v_{1,2} = \frac{-7 \pm \sqrt{49 + 4\cdot 60}}{2} = \frac{-7 \pm 17}{2} = \begin{cases} \boxed{5} \\ -12 \end{cases}$$

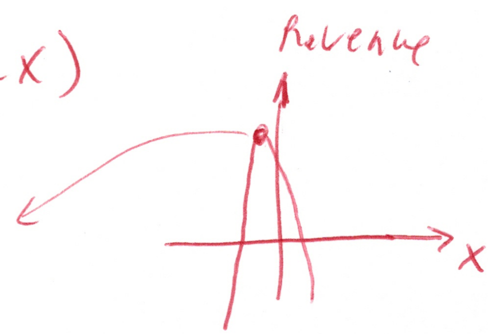
$\sqrt{289}$

4. **Question type: You need to make a formula**

You run a canoe-rental business on a small river in Ohio. You currently charge \$12 per canoe and average 36 rentals a day. An industry journal says that, for every fifty-cent increase in rental price, the average business can expect to lose two rentals a day. Use this information to attempt to maximize your income. What should you charge?

Price hike	Rental price	# of rentals	Total income / revenue
0	12	36	$12 \times 36 = \$432$
1	$12 + 1 \times 0.5$	$36 - 1 \times 2$	$12.5 \times 34 = \$425$
2	$12 + 2 \times 0.5$	$36 - 2 \times 2$	$13.0 \times 32 = \$416$
3	$12 + 3 \times 0.5$	$36 - 3 \times 2$	$13.5 \times 30 = \$405$
x	$12 + x \cdot 0.5$	$36 - x \cdot 2$	$(12 + 0.5x)(36 - 2x)$

$\Rightarrow$ Revenue $= (12 + 0.5x) \cdot (36 - 2x)$



① Desmos plot: $(-3, 441)$

That means: If I give $\underline{\underline{-3}}$ increases $\Rightarrow 12 - 3 \times 0.5 = \boxed{\$10.}$

my Revenue will be maximized $\boxed{\$441.}$

② Analytic way:

$$(12 + 0.5x)(36 - 2x) = 432 - 24x + 18x - x^2 = \underline{-x^2 - 6x + 432}$$

$$h = -\frac{b}{2a} = \frac{-(-6)}{2 \cdot (-1)} = \boxed{-3} \Rightarrow 12 - 3 \times 0.5 = \$\boxed{10.5}$$

$$k = \text{plug in } (-3) = -(-3)^2 - 6(-3) + 432 = \$\boxed{441}$$

5. **Question type: Equations set**

Find the solutions of:

$$\begin{cases} y = -x^2 + 4x - 3 & (*) \\ x + y = 1 & (**) \end{cases}$$

$(**)$ $x+y=1 \Rightarrow y=1-x$

substituting: $(*)$ $1-x=-x^2+4x-3$

$$-x^2+5x-4=0$$

$$x_{1,2}=\frac{-5\pm\sqrt{25-16}}{-2}=\frac{-5\pm\sqrt{9}}{-2}=\frac{-5\pm3}{-2}=\begin{cases}1\\4\end{cases}$$

Now, you solve.

$\boxed{x=1, y=0}$ or $\boxed{x=4, y=-3}$

check:

$x+y \overset{?}{=} 1$ | $x+y=1$
$1+0=1$ ✓ | $4-3 \overset{?}{=} 1$ ✓

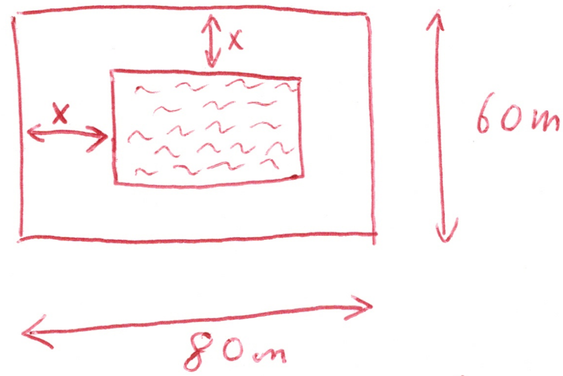
$0 \overset{?}{=} -1^2+4\cdot1-3$ | $-3 \overset{?}{=} -(4)^2+4\cdot4-3$

$0 \overset{?}{=} -1+4-3$ | $-3 \overset{?}{=} -16+16-3$

✓ | $-3 \overset{?}{=} -3$ ✓

6. Question from the book (page 347) - Make an equation and solve

A rectangular lawn is 60m by 80m. The center part of the lawn is torn up to install a pool, leaving a strip of lawn of uniform width around the pool. The area of the pool is $\frac{1}{6}$ of the old lawn area. How wide is the strip of the lawn?



Pool Area $= (80-2x)\cdot(60-2x)$

Also:
Pool Area $= \frac{1}{6}(80\cdot 60) = 800$

$$(80-2x)(60-2x) = 800$$

$$4800 - 280x + 4x^2 = 800$$

$$x^2 - 70x + 1000 = 0 \Rightarrow x_{1,2} = \frac{70 \pm \overbrace{\sqrt{70^2 - 4\cdot 1000}}^{\sqrt{900}}}{2} = \frac{70 \pm 30}{2} = \begin{cases} 50 \times \\ \boxed{20\text{ m}} \end{cases}$$

Doesn't make sense: Larger than lawn!!

7. Question from the book (page 349, Q 8) - Make an equation and solve

Find three consecutive integers such that the square of the first plus the product of the other two is 46.

$$n,\ n+1,\ n+2$$

$$n^2 + (n+1)(n+2) = 46$$

$$n^2 + n^2 + 3n + 2 = 46$$

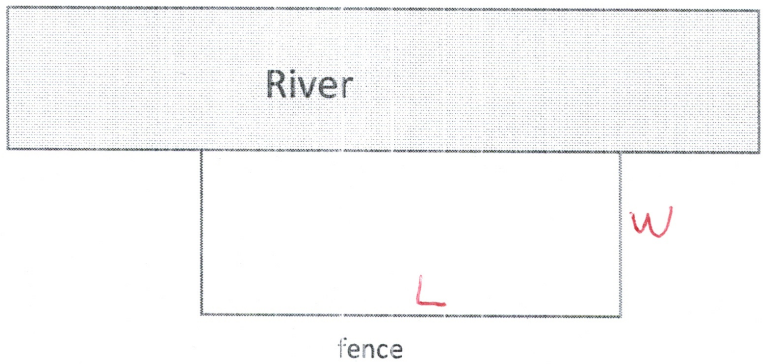
$$2n^2 + 3n - 44 = 0 \Rightarrow n_{1,2} = \frac{-3 \pm \overbrace{\sqrt{9 + 4\cdot 2\cdot 44}}^{\sqrt{361}}}{4} = \frac{-3 \pm 19}{4} = \begin{cases} \boxed{4} \\ -5\frac{1}{2} \times \end{cases}$$

not integer.

$$\boxed{4, 5, 6}$$

8. Question rom the book (page 411, question 29) - Optimization

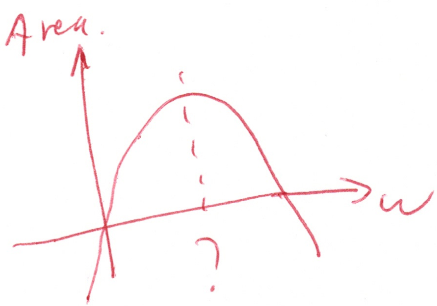
A farmer wants to build a rectangular fence near a river, and he will use 120 ft of fencing. The side next to the river will not be fenced. What is the area of the largest region that can be enclosed?



$$\left.\begin{array}{l} 2w + L = 120 \\ \text{Area} = w \cdot L \end{array}\right\} \rightarrow L = 120 - 2w$$

$$\text{Area} = w \cdot (120 - 2w) = -2w^2 + 120w$$

$$\text{vertex} - h = \frac{-b}{2a} = \frac{-120}{2 \cdot (-2)} = \boxed{30}$$



$w = 30$

$L = 60$

$$\text{Area} = 30 \cdot 60 = \boxed{1800}\ ft^2$$